SATURATION CLASSES FOR A SEQUENCE OF CONVOLUTION OPERATORS WITH LIMITED OSCILLATION

BY

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ABSTRACT

Saturation classes for the sequence $K_n(f, x) = \int f(x - t)\, d\mu_n(t)$ of linear operators where $K_n(f, x)$ is of the limited oscillation type, that is, $\mu_n(t)$ is monotonic for $t \notin [-A\sigma_n, A\sigma_n]$, $\sigma_n = o(1)$, $n \to \infty$, and $\sigma_n^{2m} = \int t^{2m}\, d\mu_n(t)$, are obtained. Examples of applications to some sequences of non-positive operators are given.

1. Introduction

A sequence of linear operators $K_n(f, x)$ from a Banach space E to itself is an approximation process if $\| K_n(f, \cdot) - f(\cdot) \|_E = o(1)$ for all $f \in E$. An approximation process $K_n(f, x)$ is called saturated with saturation (Favard) class $\mathscr{R}$ if for $f \in E$

$$(1.1) \qquad \| K_n(f, \cdot) - f(\cdot) \|_E = \begin{cases} O(\chi_n) & \text{if and only if } f \in \mathscr{R} \\ o(\chi_n) & \text{if and only if } f \in \mathscr{R}_0 \end{cases}$$

where $\mathscr{R}_0$ is a trivial subspace of $\mathscr{R}$ (generally either the zero space or a finite-dimensional space). The class of operators we shall treat will be the convolution operators

$$(1.2) \qquad K_n(f, x) = \int f(x - t) d\mu_n(t)$$

with domain $(-\infty, \infty)$ or $(-\pi, \pi)$ where, in the latter case, we assume f to be 2π periodic.

Saturation classes for many operators of type (1.2) are known, such as [1],

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[3], [7], and [8]. Most of the applications to the above are for the case in which μ_n is monotonic. Recently G. Freud and I investigated the rate of approximation for $f \in C$ where monotonity of $\mu_n(t)$ was replaced by limiting the interval in which the trend is changed to $(-A\sigma_n, A\sigma_n)$ where $\sigma_n = o(1)$. (Refer to [4].) It is the goal of this paper to find the saturation class for such operators and to give some applications that are not covered by known theorems.

For the proof of the saturation result we use distributions and weak limits, first used by H. Buchwalter [2] and then by many others for proving various saturation results. (Consult [1] and [7] for references.)

2. The saturation result

The main result of the paper is the following theorem.

THEOREM 2.1. *Suppose:*

(i) $K_n(f,x) = \int f(x-t)d\mu_n(t)$ *where the integral is* $(-\infty,\infty)$ *or* $(-\pi,\pi)$ *in which case* $f(x)$ *and* $\mu_n(t)$ *are* 2π *periodic*;

(ii) $\int |d\mu_n(t)| \leqq M$, *M independent of n*;

(iii) $\int t^i d\mu_n(t) = \alpha_i \sigma_n^{2k} + o(\sigma_n^{2k})$, $n \to \infty$, $1 \leqq i \leqq 2k$. $\Sigma_{i=1}^{2k} |\alpha_i| \neq 0$, $\int d\mu_n(t) = 1 + o(\sigma_n^{2k})$ *and* $\sigma_n = o(1)$, $n \to \infty$;

(iv) $\int |t|^{2k+\beta} d\mu_n(t) = o(\sigma_n^{2k})$ *for some* $\beta > 0$;

(v) $\mu_n(t)$ *is either increasing or decreasing for* $|t| > A\sigma_n$ *for some A independent of n.*

Then, for $E = L_p$, $1 \leqq p < \infty$ *or* $C_0, f \in E$,

(2.1) $$\| K_n(f,\cdot) - f(\cdot) \| = O(\sigma_n^{2k}),\ n \to \infty \text{ if and only if } f \in \mathscr{R} \subset E$$

and

(2.2) $$\| K_n(f,\cdot) - f(\cdot) \| = o(\sigma_n^{2k}),\ n \to \infty \text{ if and only if } f \in \mathscr{R}$$

and

$$\sum_{\pi=1}^{2k} (-1) \frac{\alpha_i}{i!} f^{(r)}(x) \equiv P(D)f = DP_1(D)f = 0 \text{ where}$$

(2.3) $$\mathscr{R} = \begin{cases} P(D)f \in L_p, & \text{if } E = L_p,\ 1 < p < \infty \\ P(D)f \in L_\infty, & \text{if } E = C_0 \\ P_1(D)f \in BV & \text{if } E = L_1. \end{cases}$$

REMARK 2.2. The conclusion of Theorem 2.1 remains true if (iii) and (iv) are replaced by the weaker assumptions:

(iii)′ For some $\eta > 0$, $\int_{|t|<\eta} t^i d\mu_n(t) = \alpha_i \sigma_n^{2k} + o(\sigma_n^{2k})$, for $n \to \infty$, $1 \leqq i \leqq 2k$, $\sum_{i=1}^{2k} |\alpha_i| \neq 0$, and $\int d\mu_n(t) = 1 + o(\sigma_n^{2k})$.

(iv)′ For $\eta > 0$ of condition (iii)′ and some $\beta > 0$, $\int_{|t|<\eta} |t|^{2k+\beta} d\mu_n(t) = o(\sigma_n^{2k})$ and $\int_{|t|<\eta} d\mu_n(t) = o(\sigma_n^{2k})$.

With these conditions, the proof is similar but longer. As an example for $k = 1$ that (iii)′ and (iv)′ are satisfied but not (iii) and (iv), we have the following:

$$K_n(f, x) = \int_{-\infty}^{\infty} \left[\frac{nK(\gamma)(1 - n^{-\delta})}{1 + n^{\gamma}|x - t|^{\gamma}} + \frac{n^{1-\delta}K(\beta)}{1 + n^{\beta}|x - t|^{\beta}} \right] f(t) dt$$

where $\gamma > 3$, $2 < \beta < 3$, $\delta > 3^{-\beta}$, and $K(\alpha)$ is given by

$$K(\alpha) = \left[\int_{-\infty}^{\infty} \frac{dt}{1 + |t|^{\alpha}} \right]^{-1}.$$

Another (more primitive) example is $K_n(f, x) = \frac{1}{2} f(x + 1/n) + \frac{1}{2} f(x - 1/n) + (1/n^3) f(x + n)$. Since the examples above are somewhat artificial, the applicable conditions seem to be those stated in Theorem 2.1.

REMARK 2.3. In general $P(D)f = 0$, $f \in E$ implies $f = 0$ except in the case of 2π periodic functions and the domain $(-\pi, \pi)$, in which case solutions as $\sin 2l\pi x$ and $\cos 2l\pi x$ are possible.

REMARK 2.4. For a sequence of positive operators our theorem is as follows.

COROLLARY 2.4. *Let* $\mu_n(t)$ *be monotonic,*

$$\int d\mu_n(t) = 1 + o(\sigma_n^2), \quad \int t d\mu_n(t) = \alpha_1 \sigma_n^2 + o(\sigma_n^2), \quad \int t^2 d\mu_n(t) = \sigma_n^2$$

and $\int |t|^{2+\beta} d\mu_n(t) = o(\sigma_n^2)$, *then for* $E = L_p$, $1 \leqq p < \infty$ or $E = C_0$ $\| K_n(f, \cdot) - f(\cdot) \|_E = O(\sigma_n^2)$ *if and only if* $f \in \mathscr{R} \subset E$, *where*

$$\mathscr{R} = \begin{cases} f'' - \alpha_1 f' \in L_p & \text{if } E = L_p, 1 < p < \infty \\ f'' - \alpha_1 f' \in L_{\infty} & \text{if } E = C_0 \\ f' - \alpha_1 f \in BV & \text{if } E = L_1. \end{cases}$$

This corollary is partially known though not precisely under the present conditions.

3. Proof of the main result

For the proof of Theorem 2.1 we need the following main lemma.

LEMMA 3.1. *Suppose conditions (i)–(v) of Theorem 2.1 are satisfied and let* $K^*(\psi,x) = \int \psi(x+t)d\mu_n(t)$, *then for* $E^* = L_p$, $1 < p \leqq \infty$, *or* $E^* = BV$ *we have for* $\psi \in \mathscr{D}$ *(where* $\mathscr{D}$ *is a Schwartz space of test functions)*

$$(3.1) \qquad \left\| \sigma_n^{-2k}(K_n^*(\psi,\cdot) - (\cdot)) - \sum_{i=1}^{2k} \frac{\alpha_i}{i\,!}\psi^{(i)}(\cdot)\right\|_{E^*} = o(1), \qquad n \to \infty .$$

PROOF. Recalling that $\int d\mu_n(t) = 1 + o(\sigma_n^{2k})$, $n \to \infty$, we have

$$\left\| \sigma_n^{-2k}\left[\int \psi(x+t)d\mu_n(t) - \psi(x)\right] - \sum_{i=1}^{2k} \frac{\alpha_i}{i!}\psi^{(i)}(x)\right\|_{E^*}$$

$$\leqq \left\| \sigma_n^{2k}\left[\int_{|t| \leqq \eta} \{\psi(x+t) - \psi(x)\}d\mu_n(t)\right] - \sum_{i=1}^{2k} \frac{\alpha_i}{i!}\psi^{(i)}(x)\right\|_{E^*}$$

$$+ \sigma_n^{-2k}\left\| \int_{|t|>\eta} \{\psi(x+t) - \psi(x)\}d\mu_n(t)\right\|_{E^*} + o(1)\,\|\psi(\cdot)\|_{E^*}$$

$$\equiv I_1(n) + I_2(n) + I_3(n).$$

Since $\psi \in \mathscr{D}$, $\|\psi(\cdot)\|_{E^*} < \infty$, $I_3(n) = o(1)$, $n \to \infty$. Also for $|t| > \eta$, $\mu_n(t)$ is monotonic and therefore for any fixed η,

$$I_2 = \sigma_n^{-2k}2\|\psi(\cdot)\|_{E^*}\left| \int_{|t|>\eta} d\mu_n(t)\right| = \sigma_n^{-2k}2\|\psi(\cdot)\|_{E^*}\frac{1}{\eta^{2k+\beta}}\left| \int_{|t|>\eta} |t|^{2k+\beta}d\mu_n(t)\right|$$

$$= o(1).$$

We shall estimate I_1 now using Taylor's formula.

$$I_1(n) \leqq \left\| \sigma_n^{-2k}\int_{|t|\leqq\eta} \sum_{i=1}^{2k} \frac{t^i}{i!}\psi^{(i)}(x)d\mu_n(t) - \sum_{i=1}^{2k} \frac{\alpha_i}{i!}\psi^{(i)}(x)\right\|_{E^*}$$

$$+ \sigma_n^{-2k}\left\| \int_{|t|\leqq\eta} \frac{t^{2k}}{(2k)!}\varepsilon(x,t)d\mu_n(t)\right\|_{E^*} \equiv \mathscr{I}_1(n) + \mathscr{I}_2(n).$$

To estimate $\mathscr{I}_1(n)$, we write

$$\mathscr{I}_1(n) \leqq \sum_{i=1}^{2k} \frac{\|\psi^{(i)}(x)\|_{E^*}}{i!}\left| \sigma_n^{-2k}\int_{|t|\leqq\eta} t^i d\mu_n(t) - \alpha_i\right| .$$

Since

$$\left| \int_{|t|>\eta} t^i d\mu_n(t)\right| \leqq \frac{1}{\eta^{2k+\beta-i}}\int_{|t|>\eta} |t|^{2k+\beta}d\mu_n(t) = o(\sigma_n^{2k}),$$

we have

$$\int_{|t|\leqq\eta} t^i d\mu_n(t) = \int t^i d\mu_n(t) + o(\sigma_n^{2k}) = \alpha_i\sigma_n^{2k} + o(\sigma_n^{2k})$$

and therefore $\mathscr{I}_1(n) = o(1)\ n \to \infty$. We estimate $\mathscr{I}_2(n)$, and choose $\eta > 0$ separately for various E^*. For $E^* = L_\infty(-\infty,\infty)$ it is enough to choose $\eta > 0$ such that $|\varepsilon(x,t)| < \varepsilon$ for $|t| < \eta$ and all x, which implies

$$\begin{aligned}\mathscr{I}_2(n) &= \frac{\sigma_n^{-2k}}{(2k)!}\left|\int_{|t|\leqq\eta} t^{2k}\varepsilon(x,t)d\mu_n(t)\right| \\ &\leqq \frac{\sigma_n^{-2k}}{(2k)!}\int_{|t|\leqq A\sigma_n} t^{2k}|\varepsilon(x,t)|\,|d\mu_n(t)| + \left|\frac{\sigma_n^{-2k}}{(2k)!}\int_{A\sigma_n<|t|<\eta} t^{2k}|\varepsilon(x,t)|\cdot d\mu_n(t)\right| \\ &\leqq \frac{\sigma_n^{-2k}}{2k!}A^{2k}\sigma_n^{2k}\varepsilon M + \varepsilon\frac{\sigma_n^{-2k}}{(2k)!}\left|\int_{A\sigma_n<|t|<\eta} t^{2k}d\mu_n(t)\right|.\end{aligned}$$

Since

$$\left|\int_{A\sigma_n<|t|\leqq\eta} t^{2k}d\mu_n(t)\right| \leqq \left|\int_{A\sigma_n<|t|} t^{2k}d\mu_n(t)\right| \leqq M_1\sigma_n^{2k},$$

we have $\mathscr{I}_2(n) \leqq M_2\varepsilon$, where M_2 is independent of ε.

For $E^* = L_p$, $1 < p < \infty$, we recall that if the support of ψ is in $[-B,B]$, $\varepsilon(x,t) = 0$ for $x \notin [-B-2\eta, B+2\eta]$ and $t \in [-\eta,\eta]$. Therefore, dividing the interval $|t| \leqq \eta$ to $|t| \leqq A\sigma_n$ and $A\sigma_n < |t| \leqq \eta$ as before, we have

$$\begin{aligned}\left\|\sigma_n^{-2k}\int_{|t|\leqq\eta}\frac{t^{2k}}{(2k)!}\varepsilon(x,t)d\mu_n(t)\right\|_p &= \left[\int_{B-2\eta}^{B+2\eta}\left\{\frac{\sigma_n^{-2k}}{(2k)!}\left|\int_{|t|\leqq A\sigma_n}\varepsilon(x,t)t^{2k}d\mu_n(t)\right|\right\}^p\right]^{1/p} \\ &+ \left[\int_{B-2\eta}^{B+2\eta}\left\{\frac{\sigma_n^{-2k}}{2k!}\int_{A\sigma_n<|t|\leqq\eta}|\varepsilon(x,t)|t^{2k}d\mu_n(t)\right\}^p\right]^{1/p} \\ &\leqq \frac{\varepsilon}{(2k)!}(A^{2k}+M_1)(2B+4\eta)^{1/p}.\end{aligned}$$

For $E^* = BV$ we recall that the support of $\varepsilon(x,t)$ is $[-A-2\eta, A+2\eta]$ and that, since $\psi \in \mathscr{D}$, we have

$$\psi'(x+t) - \psi'(x) = \sum_{i=1}^{2k}\frac{t^i}{i!}\psi^{(i+1)}(x) + \frac{t^{2k}}{(2k)!}\frac{d}{dt}\varepsilon(x,t).$$

Therefore for η small enough $(d/dx)\varepsilon(x,t)| < \varepsilon$ for $|t| \leqq \eta$ and any x. We now have

$$\begin{aligned}&\left\|\frac{\sigma_n^{-2k}}{(2k)!}\left\{\int_{|t|\leqq A\sigma_n} + \int_{A\sigma_n<|t|\leqq\eta}\right\}\varepsilon(x,t)d\mu_n(t)\right\|_{BV} \\ &\leqq \frac{\sigma_n^{-2k}}{(2k)!}\left[\int\int_{|t|\leqq A\sigma_n} t^{2k}\left|\frac{d}{dx}\varepsilon(x,t)\right|\,|d\mu_n(t)|\right] \\ &\quad + \int\int_{A\sigma_n<|t|\leqq\eta} t^{2k}|d\mu_n(t)|\,\left|\frac{d}{dx}\varepsilon(x,t)\right| \leqq L\varepsilon.\end{aligned}$$

PROOF OF THEOREM 2.1 We shall show first that $f \in \mathscr{R}$ implies

$$\| K_n(f,\cdot) - f(\cdot) \|_E = O(\sigma_n^{2k}) \text{ and } = o(\sigma_n^{2k})$$

in the case $P(D)f = 0$. For $\psi \in \mathscr{D}$, using Lemma 3.1, we have for $E = L_p$, $1 < p$. $< \infty$, and $E = C$:

$$\left(\frac{1}{\sigma_n^{2k}}[K_n(f,\cdot) - f(.)], \psi(\cdot)\right) = \left(f(\cdot), \frac{1}{\sigma_n^{2k}} K_n^*(\psi,\cdot) - \psi(\cdot)\right)$$

$$(3.2) \quad = \langle f(\cdot), P(-D)\psi\rangle + o(1)\| f \|_E = \langle P(D)f(\cdot), \psi(\cdot)\rangle + o(1)\|f\|_E.$$

For L_p $1 < p < \infty$, we recall that $\psi \in \mathscr{D}$ is dense in L_q, $1 \leqq q < \infty$, and therefore the result is immediate. For $E = C$ we use $\mathscr{D}$ dense in L_1 and write

$$\| K_n(f,\cdot) - f(\cdot) \|_C \leqq \| K_n(f,\cdot) - f(\cdot) \|_{L\infty} \leqq \sigma_n^{2k} \| P(D)f(\cdot) \|_L.$$

For $E = L$, we interpret $\langle P(D)f(\cdot), \psi(\cdot)\rangle = \int \psi(x)d[P_1(D)f(x)]$ and therefore

$$\left\langle \left| \frac{1}{\sigma_u^{2k}}[K_n(f,\cdot) - f(\cdot)], \psi(\cdot)\right\rangle\right| = \left| \int \psi(x)d(P_1(D)f(x)) \right| + o(1)\|f\|$$
$$\leqq \| \psi \|_C \cdot \| P_1(D)f(\cdot) \|_{BV} \leqq \| \psi(\cdot) \|_{L\infty} \cdot \| P_1(D)f(\cdot) \|_{BV}.$$

Since $K_n(f,x) - f(x)$ is continuous (when $P_1(D)f \in BV$),

$$\sup_{\|\phi\|_{L\infty} \leqq 1} \langle K_n(f,x) - f(x), \phi(x)\rangle = \sup_{\|\phi\|_C \leqq 1} \langle K_n(f,x) - f(x), \phi(x)\rangle,$$

and since $\mathscr{D}$ is dense in C_0, we have

$$\left\| \frac{1}{\sigma_n^{2k}} K_n(f,\cdot) - f(\cdot) \right\|_{L_1} \leqq M.$$

Now we shall prove that $f \in \mathscr{R}$ is necessary for

$$\| K_n(f,\cdot) - f(\cdot) \|_E = O(\sigma_n^{2k}) \ n \to \infty.$$

We recall that for $1 < p \leqq \infty$, $L_p \simeq L_q^*$, $q^{-1} + p^{-1} = 1$, and that the closed unit sphere of the dual of a Banach space is weak* compact. Therefore for $E = L_p$, $1 < p < \infty$, or $E = C_0 \subset L_\infty$, there exists $\{n_i\}$ and $g \in L_p$, $1 < p \leqq \infty$, such that

$$(3.3) \quad \lim_{i \to \infty} \int \sigma_{n_i}^{-2k}[K_{n_i}(f,x) - f(x)]\psi(x)dx = \int g(x)\psi(x)\,dx \text{ for all } \psi \in L_1$$

and in particular for all $\psi \in \mathscr{D}$. Combining (3.2) and (3.3), we obtain $\int g(x)\psi(x)dx$ $= \int f(x)P(-D)\psi(x)dx$ or, in the distribution sense, $P(D)f(x) = g(x)$ in $\mathscr{D}'$, which means that an equivalent to $f(x)$ (in the L_p sense) has $2k - 1$ derivatives so that $P_1(D)f$ is absolutely continuous locally and $P(D)f \in L_p$.

To prove our theorem for $E = L_1$, we recall that $f \in L_1$ implies $\phi(u) = \int_{\infty}^{u} f(x)dx$ where the integral exists and $\phi(u) \in NBV$, and moreover $\| \phi \|_{BV} = \| f \|_{L_1}$. Recalling that $NBV = C_0^*$ and using the compactness of the closed unit sphere in a dual of a Banach space in the weak* sense, we have

$$(3.4) \qquad \lim_{i \to \infty} \int \sigma_{n_i}^{-2k}[K_{n_i}(f,x) - f(x)]\psi(x)dx = \int \psi(x)dg(x)$$

for $\psi(x) \in C_0$, in particular for $\psi \in \mathscr{D}$. Following the above, $g(x) = P_1(D)f \in BV$. Obviously now if $\| K_n(f,x) - f(x) \| = o(\sigma_n^{-2k})$, we have $P(D)f = 0$.

As a corollary of the proof, we can derive a local version of Theorem 2.1.

COROLLARY 3.2. *If assumptions* (i)–(v) *of Theorem* 2.1 *are satisfied, then for* $f \in E$

$$(3.5) \quad \| K_n(f,\cdot) - f(\cdot) \|_{E(a,b)} = \begin{cases} O(\sigma_n^{2k}) & \text{if and only if } f \in \mathscr{R}[a,b] \\ o(\sigma_n^{2k}) & \text{if and only if } P(D)f = 0 \text{ for } a < x < b \end{cases}$$

where $E(c,d)$ *is* $L_p(c,d)$ *for* $1 \leqq p < \infty$ *or* $BC[c,d]$ *for all* (c,d) *such that* $[c,d] \subset [a,b]$

$$\mathscr{R}[a,b] = \begin{cases} P(D)f \in L_p(\alpha,\beta) \text{ if } E = L_p \\ P(D)f \in L_\infty(\alpha,\beta) \text{ if } E = C \\ P_1(D)f \in BV(\alpha,\beta) \text{ if } E = L_1 \quad \text{for } a < \alpha < \beta < b. \end{cases}$$

The proof follows that of Theorem 2.1 using, instead of the space $\mathscr{D}$, the space $\mathscr{D}(a,b)$, that is, the C^∞ functions with compact support in (a,b).

4. Application of combinations to δ functions

Let $0 < \gamma < 1$ $(1-\gamma)f(x - (\gamma/n) + \gamma f(x + (1-\gamma)/n)$ approximates $f(x)$, and the saturation theorem is given as the following corollary of Theorem 2.1.

THEOREM 4.1. *For* $f \in E$, $E = L_p$, $1 \leqq p < \infty$, *or* $E = C_0$, *we have*

$$(4.1) \quad \left\| (1-\gamma)f\left(\cdot - \frac{\gamma}{n}\right) + \gamma f\left(\cdot + \frac{1-\gamma}{n}\right) - f(\cdot) \right\|_E = O(n^{-2})$$

if and only if $f \in \mathscr{R}$ *where* $P(D)f(x) = f''(x)$.

5. Application to inversion of convolution transforms

Saturation classes for inversion of convolution transforms of variation-diminishing kernels were determined by the author in [3]. That saturation result as well as others that have a non-positive kernel will be derived as corollaries of Theorem 2.1.

Recall [6]

(5.1) $$E_m(s) = e^{b_m s} \prod_{k=m+1}^{\infty} (1 - s/a_k)e^{s/a_k}, \qquad m = 0, 1, \cdots$$

where b_m and a_k are real $\Sigma a_k^{-2} < \infty$,

(5.2) $$G_m(t) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} [E_m(s)]^{-1} e^{st} ds, \qquad m = 0, 1, \cdots,$$

(5.3) $$f(x) = \int_{-\infty}^{\infty} G(x - t)f(t)dt \text{ where } G(t) = G_0(t)$$

and

(5.4) $$P_m(D)F(x) = \int_{-\infty}^{\infty} G_m(x - t)f(t)dt.$$

For $\sigma_m^2 = \Sigma_{m+1}^{\infty} a_k^{-2}$ $b_m = \alpha\sigma_m^2 + o(\sigma_m^2)$ and $E = L_p$ $1 \leqq p < \infty$ or C_0, we have

(5.5) $$\| P_m(D)F(\cdot) - f(\cdot) \|_E = O(\sigma_m^2) \text{ if and only if } f \in \mathscr{R}$$

where $P(D)f(x) = f''(x) - \alpha f'(x)$. This result was achieved in [3], though here the local analog also follows.

A combination of $P_m(D)f(x)$ for different m may yield a higher rate of convergence and different saturation classes. However, these results depend on the particular sequence of $\{a_k\}$. The results below can be followed by results for many transforms: for instance, examples $(A), \cdots (H)$, but not (I), given by Hirschman and Widder [6], to the real inversion formulas of the type discussed above.

First we recall [5, p. 278] for $G_m(t)$ given by (5.2) we have

(5.6) $$\int_{-\infty}^{\infty} t^{2n} G_m(t)dt \leqq K(n)\sigma_m^{2n}.$$

We recall also [6, p. 94] that $G_m(t) - aG_{m+r}(t)$ has at most two changes of sign.

For $\Sigma_{k=m+1}^{\infty} a_k^{-3} = o(\sigma_m^3)$ $m \to \infty$, $G_m(t) - aG_{lm+r}(t)$ has exactly two changes of sign for $|t| < A\sigma_m$, where A depends only on a, which follows the asymptotic estimate [6, p. 140]. Therefore, for a combination $\Sigma_{j=1}^{2k} \beta j G_{mj}(t) = H_m(t)$, for which $\Sigma_{i=1}^{2k} \beta_i = 1$ and $\Sigma_{k=m+1}^{\infty} a_k^{-3} = o(\sigma_m^3)$, we only have to show, in order to use Theorem 2.1, that $\int t^i H_m(t)dt = \alpha_i \sigma_m^{2k+2} + o(\sigma_m^{2k+2})$ for $1 \leqq i \leqq 2k + 1$. We can choose $b_m = 0$ or $\alpha_1 = 0$. Straightforward calculation yields for $\mu_i(m) \equiv \int t^i G_m(t)dt$, $\mu_2(m) = \Sigma_{k=m+1}^{\infty} a_k^{-2}$, $\mu_3(m) = 2 \Sigma_{k=m+1}^{\infty} a_k^{-3}$ and $\mu_4(m) = 3(\Sigma_{k=m+1}^{\infty} a_k^{-2})^2 + 6 \Sigma_{k=m+1}^{\infty} a_k^{-4}$.

In [6, examples (A) through (H)], $H_m(t)$ can be calculated for $P(D) = \sum_{r=2}^{4} \alpha_i D^i$, ($P(D)$ of Theorem 2.1). For instance, in example A, $a_k = k$, $H_m(t) = 2G_{2m}(t) - G_m(t)$. We can write

$$\begin{aligned}\int t^2 H_m(t)dt &= 2\mu_2(2m) - \mu_2(m) = 2 \sum_{k=2m+1}^{\infty} \frac{1}{k^2} - \sum_{k=m+1}^{\infty} \frac{1}{k^2} \\ &= 2 \sum_{k=2m+1}^{\infty} \frac{1}{k(k+1)} - \sum_{k=m+1}^{\infty} \frac{1}{k(k+1)} + 2 \sum_{k=2m+1}^{\infty} \frac{1}{k^2(k+1)} \\ &\quad - \sum_{k=m+1}^{\infty} \frac{1}{k^2(k+1)} \\ &= \frac{2}{2m+1} - \frac{1}{m+1} + \frac{1}{(2m+1)^2} - \frac{1}{2}\frac{1}{(m+1)^2} + O\left(\frac{1}{m^3}\right). \\ &= \frac{1}{4}\frac{1}{m^2} + O\left(\frac{1}{m^3}\right).\end{aligned}$$

Similarly

$$\begin{aligned}\int t^3 H_m(t)dt &= \frac{-1}{2}\frac{1}{m^2} + O\frac{1}{(m^3)}, \quad \text{and} \\ \int t^4 H_m(t)dt &= 2\cdot 3\frac{1}{4m^2} - 2\cdot 3\frac{1}{m^2} + O\left(\frac{1}{m^3}\right) = \frac{-9}{2}\cdot\frac{1}{m^2} + O\left(\frac{1}{m^3}\right).\end{aligned}$$

As a corollary we have the following.

COROLLARY 5.1. *Let $f \in E$ (L_p, $1 \leqq p < \infty$, or C_0) and let $F(x)$ be defined by* (5.3), (5.2), *and* (5.1) *where $b_m = 0$ and $a_k = k$, then*

$$\| 2P_{2m}(D)F(x) - P_m(D)F(x) - f(x) \| = O\left(\frac{1}{m^2}\right) \text{ if and only if } f \in \mathscr{R}$$

where $\mathscr{R}$ is defined in Theorem 2.1 *and $P(D) = D^2\left(\frac{1}{4} - \frac{1}{2}D - (9/2)D^2\right)$.*

6. Application to combinations of De la Vallée Poussin operators

The De la Vallée Poussin operator is defined (see, for example, Butzer and Nessel [1, p. 448], where the saturation result is also given) by

$$V_n(f;x) = \alpha_n \int_{-\pi}^{\pi} f(t)\cos^2\frac{t-x}{2}dt, \quad \alpha_n = \frac{2\cdot 4\cdot \dots \cdot 2n}{1\cdot 3\cdot \dots \cdot(2n-1)}. \tag{6.1}$$

Define

$$V_n^{(2)}(f;x) = 2V_{2n-1}(f;x) - V_{n-1}(f;x). \tag{6.2}$$

The oscillation of the kernel of $V_n^{(2)}$ is limited to $|t| < K_n^{-\frac{1}{2}}$ for some k

following [4, Sect. 5]. Also $V_n^{(2)}(1,x) = 1$, $V_n^{(2)}(t,0) = V_n^{(3)}(t^3,0) = 0$ and for any k

$$V_n^{(2)}(t^2,0) = 2V_{2n-1}(t^2,0) - V_{n-1}(t^2,0)$$

$$= 8V_{2n-1}\left(\sin^2\frac{t}{2},0\right) - 4V_{n-1}\left(\sin^2\frac{t}{2},0\right) + o\left(\frac{1}{n^2}\right) = o\left(\frac{1}{n}\right)$$

$$V_n^{(2)}(t^4,0) = 16V_n^{(2)}\left(\sin^4\frac{t}{2},0\right) + O\left(\frac{1}{n^k}\right) = 24\frac{1}{n^2} + o\left(\frac{1}{n^2}\right).$$

COROLLARY 6.1. *For $E = L_p$ or $E = C_0[-\pi,\pi]$,*

$$\| V_n^{(2)}(f,\cdot) - f \|_E = O\left(\frac{1}{n^2}\right) \text{ if and only if } f \in \mathscr{R} \text{ where } P(D)f(x) = f^{(4)}(x).$$

Similarly for $V_n^{(3)}(f,x) = (8/3)V_{4n-1}(f,x) - 2V_{2n-1}(f,x) + \frac{1}{3}V_{n-1}(f,x)$, $V_n^{(3)}(t,0) = V_n^{(3)}(t^3,0) = V_n^{(3)}(t^5,0) = 0$ and $V_n^{(3)}(t^2,0) = o(1/(n^3))$; but in thise case

$$V_n^{(3)}(t^4,0) = \alpha_4\frac{1}{n^3} + o\left(\frac{1}{n^2}\right) \text{ and } V_n^{(3)}(t^6,0) = \alpha_6\frac{1}{n^3} + o\left(\frac{1}{n^3}\right)$$

and both α_4 and α_6 are different from zero. Therefore,

$$\| V_n^{(3)}(f,\cdot) - f(\cdot) \|_{E[a,b]} = o\left(\frac{1}{n^3}\right) \text{ if and only if } f \in \mathscr{R}[a,b]$$

and

$$\alpha_4 f^{(4)}(x) + \alpha_6 f^{(6)}(x) = 0 \text{ for } x \in [a,b].$$

REFERENCES

1. P. L. Butzer and R. J. Nessel, *Fourier Analysis and Approximation*, Birkhäuser Verlag, 1971.

2. H. Buchwalter, *Saturation et distributions*, C. R. Acad. Acad. Sci. Paris **250** (1960), 2562–3564.

3. Z. Ditzian, Saturation class for inversion of convolution transforms, Proc. of the conference *On Constructive Function Theory*, Budapest, 1969.

4. Z. Ditzian and G. Freud, *Linear approximating processes with limites oscillation* (to appear).

5. Z. Ditzian and A. Jakimovski, *Inversion jump formulae for variation diminishing transforms*, Annali di Math. **81** (1969), 261–318.

6. I. I. Hirschman and D. V. Widder, *The Convolution Transform*, Princeton Univ. Press, 1955.

7. M. Kozima and G. I. Surnouchi, *On the approximation by a general singular integral*, Tôhoku Math. J. **20**, (1968), 146–169.

8. H. S. Shapiro, *Smoothing and Approximation of Functions*, Van Nostrand, 1969.

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